

Sub-ballistic behavior in the quantum kicked rotor

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Abstract

We study the resonances of the quantum kicked rotor subjected to an excitation that follows an aperiodic Fibonacci prescription. In such a case the secondary resonances show a sub-ballistic behavior like the quantum walk with the same aperiodic prescription for the coin. The principal resonances maintain the well-known ballistic behavior.

Key words: Kicked rotor; Quantum walk

The quantum kicked rotor (QKR) is considered as the paradigm of periodically driven systems in the study of chaos at the quantum level [1]. Two important characteristics of the behavior of the quantum kicked rotor are dynamical localization (DL) and the ballistic spreading of the resonances [2]. These behaviors are quite different and depend on whether the period of the kick τ is a rational or irrational multiple of 4π (in convenient units). For rational multiples, the behavior of the system is resonant and has no classical analog. For irrational multiples the average energy of the system grows in a diffusive manner for a short time and then the diffusion stops and localization appears. From the theoretical point of view the two types of values of τ determine the spectral properties of the Hamiltonian, for irrational multiples the energy spectrum is purely discrete and for rational multiples it contains a continuous part. Both quantum resonance and DL can be seen as interference phenomena, the first is a constructive interference effect and the second is a destructive one.

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We recently developed [3] a new generalized discrete time quantum walk (QW) on the line and showed that this model has a dynamical behavior analogous to that of a QKR: depending on the values of a parameter there are either quantum resonances or DL. This modified QW has been mapped into a one-dimensional Anderson model [4], as was previously done in the case of the QKR [5]. For some exceptional values of the parameter, which correspond to resonant behavior, the standard deviation grows linearly in time. We showed that the usual discrete time QW on the line becomes a particular case of resonance of the modified QW.

The concept of QW on the line is a subject that has drawn much attention in quantum computing [6]. It has been introduced in 1993 [7,8], as counterpart of the classical random walk. A classical random walk is defined in terms of the probabilities for a particle to do a step to the left or to the right but the QW is described in terms of probability amplitudes. Many classical algorithms are based on classical random walks, then it is possible that future quantum algorithms will be based on the quantum random walk. One of the most striking properties of the QW is its ability to spread over the line linearly in time as characterized by the standard deviation $\sigma(t) \sim t$, while its classical analog spreads out as the square root of time ($\sigma(t) \sim t^{1/2}$).

In this work we establish another aspect of the parallelism between the QKR and the QW following the proposal of Ribeiro *et al.* [9]. In [9] the quantum coin operator of the QW is arranged in aperiodic sequences using the Fibonacci prescription and this leads to a sub-ballistic wave function spreading ($\sigma(t) \sim t^c$, $1/2 < c < 1$). We show that the secondary resonances of the QKR, excited with the same Fibonacci prescription, have also sub-ballistic behavior like the usual QW, but the primary resonances maintain the well-known ballistic behavior. Then the parallelism with the usual QW on the line is restricted to the secondary resonances of the QKR. On the other hand Casati *et al.* [10] studied the dynamics of the QKR also kicked according to a Fibonacci sequence, but outside the resonant regime, and they found sub-diffusive behavior for small kicking strengths; in this sense our work can be considered as complementary to theirs.

The QKR Hamiltonian is

$$H = \frac{P^2}{2I} + K \cos \theta \sum_{n=1}^{\infty} \delta(t - nT) \quad (1)$$

where the external kicks occur at times $t = nT$ with n integer and T the kick period, I is the moment of inertia of the rotor, P the angular momentum operator, K the strength parameter and θ the angular position. In the angular momentum representation, $P|\ell\rangle = \ell\hbar|\ell\rangle$, the wave-vector is $|\Psi(t)\rangle = \sum_{\ell=-\infty}^{\infty} a_{\ell}(t)|\ell\rangle$ and the average energy is $E(t) = \langle\Psi|H|\Psi\rangle = \varepsilon \sum_{\ell=-\infty}^{\infty} \ell^2 |a_{\ell}(t)|^2$, where $\varepsilon = \hbar^2/2I$. Using the Schrödinger equation the quantum map is readily

obtained from the Hamiltonian (1)

$$a_\ell(t_{n+1}) = \sum_{j=-\infty}^{\infty} U_{\ell j} a_j(t_n) \quad (2)$$

where the matrix element of the time step evolution operator $U(\kappa)$ is

$$U_{\ell j} = i^{-(j-\ell)} e^{-ij^2 \varepsilon T / \hbar} J_{j-\ell}(\kappa), \quad (3)$$

J_m is the m th order cylindrical Bessel function and the argument is the dimensionless kick strength $\kappa \equiv K/\hbar$. The resonance condition does not depend on κ and takes place when the frequency of the driving force is commensurable with the frequencies of the free rotor. Inspection of eq.(3) shows that the resonant values of the scale parameter $\tau \equiv \varepsilon T / 2\hbar$ are the set of the rational multiples of 4π , $\tau = 4\pi p/q$. In what follows we assume, that the resonance condition is satisfied, therefore the evolution operator depends on κ , p and q . We call a resonance primary when p/q is an integer and secondary when it is not.

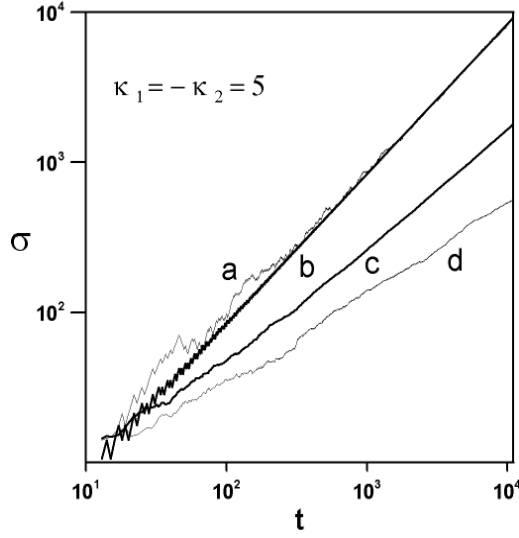


Fig. 1. Standard deviation $\sigma(t)$ as a function of time. a) random sequence for the primary resonance $p/q = 1$; b) periodical sequence for the primary resonance $p/q = 1$; c) Fibonacci sequence for the secondary resonance $p/q = 1/3$; d) random sequence for the secondary resonance $p/q = 1/3$.

With the aim to generate the dynamics of the system we consider two values of the strength parameter κ , κ_1 and κ_2 and combine the corresponding time step operators $U(\kappa_1)$ and $U(\kappa_2)$ in a large sequence that takes three different forms, namely periodic, random and quasi-periodic. In this way we generate three types of unitary evolution operators. With these operators we compute, for several thousands of T , the wave function spreading as measured by the exponent c in $\sigma(t) = \sqrt{\sum_{\ell=-\infty}^{\infty} \ell^2 |a_\ell(t)|^2} \sim t^c$.

with a given sequence of $U(\kappa)$ operators, is $a_\ell(t_n) = (-i)^\ell J_\ell \left(\sum_{j=0}^n \kappa(j) \right) = (-i)^\ell J_\ell (m_1 \kappa_1 + m_2 \kappa_2)$, where $\kappa(j)$ takes the values κ_1 or κ_2 involved in the sequence and m_1 and m_2 are the numbers of occurrences of κ_1 and κ_2 respectively, $m_1 + m_2 = n$. With the above amplitudes, the standard deviation is obtained easily as

$$\sigma(nT) = \frac{(\alpha \kappa_1 + \beta \kappa_2)}{\sqrt{2}} n \quad (4)$$

where $\alpha = m_1/n$ and $\beta = m_2/n$ are the relative weights of kicks κ_1 and κ_2 respectively. Thus $c = 1$ and, additionally, we have obtained explicitly the ballistic diffusion coefficient $D \equiv \sigma(nT)/n = (\alpha \kappa_1 + \beta \kappa_2) / \sqrt{2}$.

The QKR has an exceptional behavior when $p/q = 1/2$, called antiresonance, its characteristic being that the system returns to the initial state every two periods. In this case, given a sequence of step operators, the solution of the map eq.(2), is $a_\ell(t_n) = (-i)^\ell J_\ell \left(\sum_{j=0}^n (-i)^j \kappa(j) \right)$ and the standard deviation is $\sigma(nT) = \sum_{j=0}^n (-i)^j \kappa(j) / \sqrt{2}$. Therefore, for all sequences, the antiresonance in the case $\kappa_1 \neq \kappa_2$ has a similar behavior as the primary resonances. It is important to emphasize, that the conceptual key for the ballistic behavior of the primary resonances (and the antiresonance too) resides in the commutativity between operators $U(\kappa_1)$ and $U(\kappa_2)$. This is not the case for the secondary resonances where the commutator does not vanish.

The sub-ballistic behavior takes place for all secondary resonances in the quasi-periodic case. In Fig. 2 the exponent c is plotted as a function of the ratio p/q that identifies the secondary resonance. From this figure we can conclude that the exponent c depends on both p and q , but there is a trivial symmetry in the time step evolution operator eq.(3) when p/q is changed by $(q-p)/q$, this is the reason why *e.g.* the value of c for $p/q = 1/5$ is the same as for $p/q = 4/5$. We have also found that the dependence of c on the values of κ_1 and κ_2 is not smooth in general. The condition $\kappa_1 = \kappa_2$ corresponds to the periodic case and the expected ballistic behavior is obtained. Fig. 3 shows the cut of the surface $c(\kappa_1, \kappa_2)$ with the plane $\kappa_1 = -\kappa_2$ for $p/q = 1/3$, making clearly apparent the sub-ballistic behavior of the system for all κ . We have also studied higher moments of order four and six. The asymptotic behavior of these moments is consistent with the power-law behavior of the second moment, i.e. all the moments obtained with the Fibonacci prescription have smaller exponents than those obtained with a periodical sequence.

At this point it is possible to ask oneself if this sub-ballistic behavior has a quantum origin or if it also appears in the classical world. The classical equations of motion for the Hamiltonian eq.(1) are given by the standard map

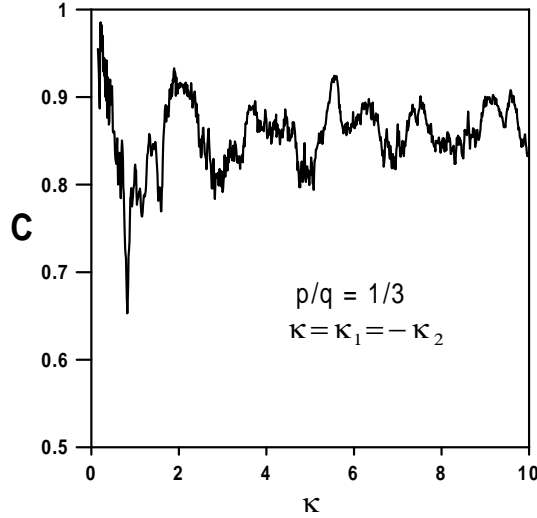


Fig. 3. The exponent c , that characterizes the power law of the standard deviation, as a function of the strength parameter κ of the kicked rotator. The dynamical evolution is obtained by the operators $U(\kappa_1)$ and $U(\kappa_2)$ following a Fibonacci succession.

[11]

$$\begin{aligned} P_{n+1} &= P_n + K \sin \theta_n \\ \theta_{n+1} &= \theta_n + P_{n+1} \mod 2\pi \end{aligned} \quad (5)$$

where n indicates the time step. In the phase space (θ_n, P_n) of the standard map a beautiful diagram is obtained where the Kolmogorov-Arnold-Moser (KAM) tori prevent diffusion in angular momentum for small values of $K < K_{cri}$. The classical model equivalent to the quantum model developed in this work is obtained when the strength parameter K takes the values K_1 and K_2 in a given sequence in correlation with the time step. We have worked numerically with the map eq.(5) in the periodic, random and aperiodic (following the Fibonacci prescription) sequences. In the periodic case the KAM surfaces are established, but in the random and aperiodic cases the KAM surfaces are broken and the classical diffusion, is established. Then we can conclude that the sub-ballistic behavior of the model is a quantum phenomenon.

Quantum resonances and DL, that were at first established in numerical and theoretical form, have been experimentally observed more than ten years ago in samples of cold atoms interacting with a far-detuned standing wave of laser

light [12,13,14]. This notable series of experiments have drawn much attention because they may be establishing both the conceptual and experimental basis of quantum computers [15,16,17,18]. Experimentally only the primary resonances of the QKR are easily observable, but recently, Kanem *et al.*[19], have observed secondary resonances. On the other hand, several experiments have been proposed [20,21,22,23] to construct models of QWs. These proposals, at first sight, have some common elements with the experimental implementation of the QKR in the recent past. Thus, in this theoretical and experimental frame where the QKR and the QW have equivalent behaviors and their experimental facilities have many elements in common, the question to be posed is: may the QKR be considered as the fundamental model of QW? More experimental and theoretical work is still necessary in order to answer this question.

In summary, we have developed an aperiodic QKR model and established a deeper equivalence between the QKR and QW. In this model we found a new sub-ballistic behavior in the secondary resonances, $\sigma(t) \sim t^c$ with $1/2 < c < 1$, where c depends on κ , p , q . The usual ballistic behavior of the QKR model is retained in the primary resonances, then these resonances are robust because they are not affected by the type of sequences of the operators $U(\kappa_1)$ and $U(\kappa_2)$, periodic, random or aperiodic. We explained this robustness by the commutativity of the U operators and we obtained the ballistic diffusion coefficient analytically. Finally, we showed numerically that the sub-ballistic behavior is a quantum interference phenomenon that has no classical analogue.

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